Basic Mathematics for Understanding Noisy Waveforms and Frequency Spectra

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Notes prepared for Yale NPA/WIDG seminar

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Part 1

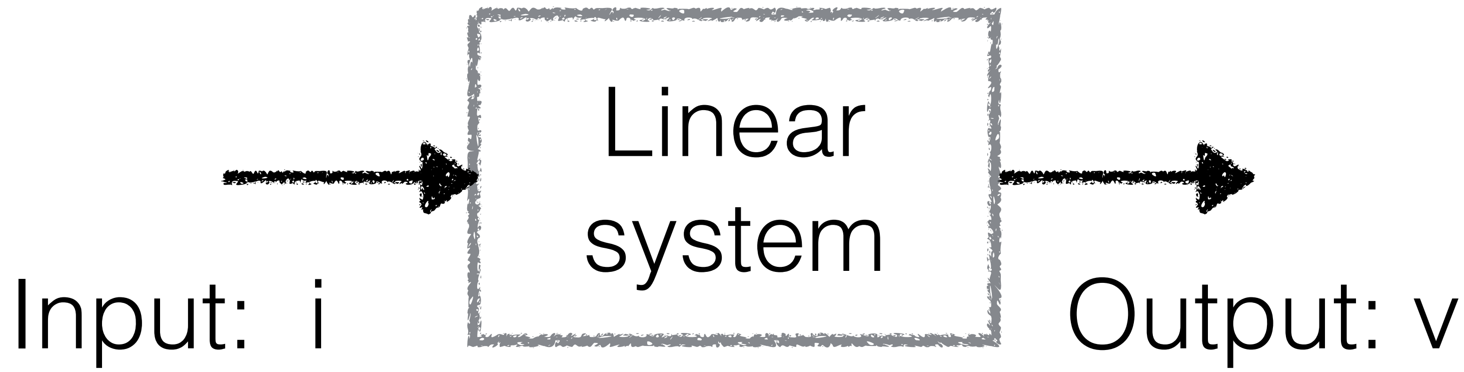
- A. basics of linear systems including examples of some common electronics and mechanical systems

- B. introduction to some needed mathematics.

This work combines three separate disciplines It is a bit heavy and requires several sittings.

The goal is to present it in an informal way so that the connections are obvious and can be later explored in a deeper way.

1) *Understanding and modeling physical systems such as electronics or mechanical systems.*
2) *Elementary Mathematical analysis and Fourier transforms.*
3) *Some elements of probability and statistics.*



- ***There are many practical mechanical or electrical systems in which the output is linearly related to the input excitation.***

- ***If the excitation is multiplied by a constant then the response is also. (homogeneity)***

- ***If there are multiple sources of excitation then the response due to each one adds linearly in the total response. (superposition).***

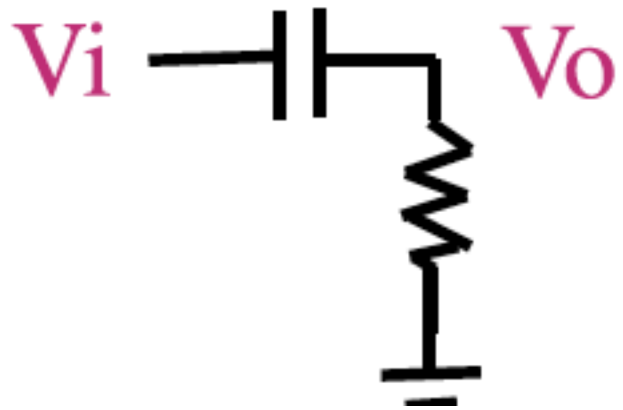
- ***The frequency of the response will be the same as the frequency of the excitation. This property allows analysis of such systems using Laplace or Fourier transforms.***

- ***This lecture is an introduction about various basic mathematics related to such systems. It is particularly useful for understanding electronics, signal processing, or mechanics of vibration.***

Outline

- ***Some simple examples of linear systems.***
- ***Impulse response function and noise waveforms.***
- ***Some methods for understanding random noise***
- ***Introduction to Fourier transforms and discrete Fourier transform.***

Example 1



High pass, differentiator

$$V_o(t)+Q/C=V_i(t)$$

$$\frac{dV_o(t)}{dt}+I(t)/C=\frac{dV_i(t)}{dt} \text{ where } I(t) \text{ is the current.}$$

$\mathrm{V}_o(t)=I(t)\times R$ therefore

$$\frac{dV_o(t)}{dt}+\frac{V_o(t)}{\tau}=\frac{dV_i(t)}{dt} \quad \text{where } \tau=RC \text{ is called the time constant.}$$

Use $\mathrm{V}_i(t)=v_i e^{i\omega t}$ for the input and $\mathrm{V}_o(t)=v_o e^{i\omega t}$ as output.

Take the real part later to get back the answer to a sinusoidal excitation.

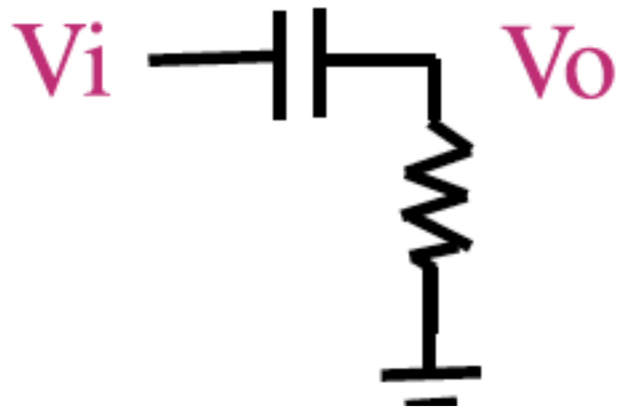
$$i\omega v_o+v_o/\tau=i\omega v_i$$

$$v_o=v_i\times\frac{i\omega}{i\omega+1/\tau}=v_i\frac{1}{\left[1+1/(\tau\omega)^2\right]^{1/2}}e^{i\theta} \text{ where } \theta=\mathrm{Arctan}(1/\tau\omega)$$

Notice that the output has lower amplitude and shifted in phase.

If $(\omega\tau)\gg 1$ then the filter just passes the input through.

For any V_i, we can calculate the frequency components (or Fourier transform), multiply by the filter function, and invert.



High pass, differentiator How to calculate for discrete data

$$V_o(t) + Q/C = V_i(t)$$

$$\frac{dV_o(t)}{dt} + i/C = \frac{dV_i(t)}{dt}$$

$$V_o = RC\left(\frac{dV_i}{dt} - \frac{dV_o}{dt}\right)$$

Notice that this will filter out DC and low frequency waveforms

For discrete data use j as the index over time bins of Δt

$$V_o(t_j) = RC\left[\frac{V_i(t_j) - V_i(t_{j-1})}{\Delta t} - \frac{V_o(t_j) - V_o(t_{j-1})}{\Delta t}\right]$$

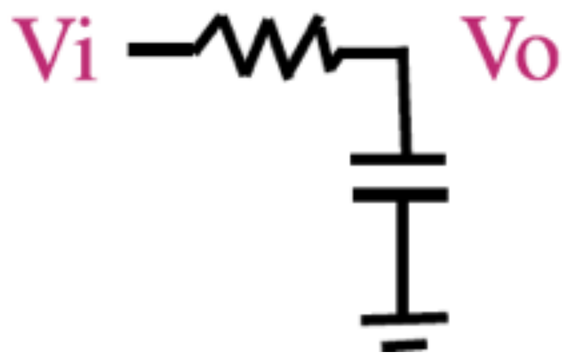
$$V_o(j) = \alpha(V_i(j) - V_i(j-1)) + \alpha V_o(j-1)$$

$$\alpha = (RC/\Delta t)/(1 + RC/\Delta t)$$

Notice that V_o changes depending on the change in the input V_i

This is why this is called the differentiator.

Example 2



Low pass, integrator.

$V_o(t) = Q/C, \qquad \text{and} \qquad V_i(t) - V_o(t) = R \cdot I(t)$

$\dfrac{dV_o(t)}{dt} = \dfrac{V_i - V_o}{\tau}$ where τ=RC

Use $V_i(t) = v_i e^{i\omega t}$ for the input and $V_o(t) = v_o e^{i\omega t}$ as output.

Take the real part later to get back the answer to a sinusoidal excitation.

$i\omega v_o + v_o/\tau = v_i/\tau$

$$v_o = v_i \times \frac{1/\tau}{i\omega + 1/\tau} = v_i \frac{1}{\left[1 + (\tau\omega)^2\right]^{1/2}} e^{i\theta} \text{ where } \theta = \text{Arctan}(\tau\omega)$$

Notice that for $\omega\tau \ll 1$ the filter just passes the input through

For discrete data with j index over time bins(Δt)

$V_o(j) = \alpha V_o(j-1) + (1-\alpha)V_i(j)$

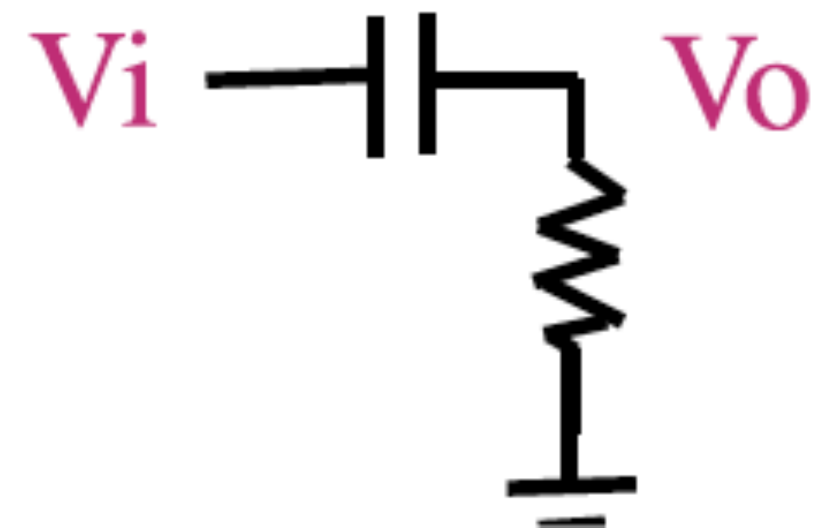
$\alpha = (RC/\Delta t)/(1 + RC/\Delta t)$

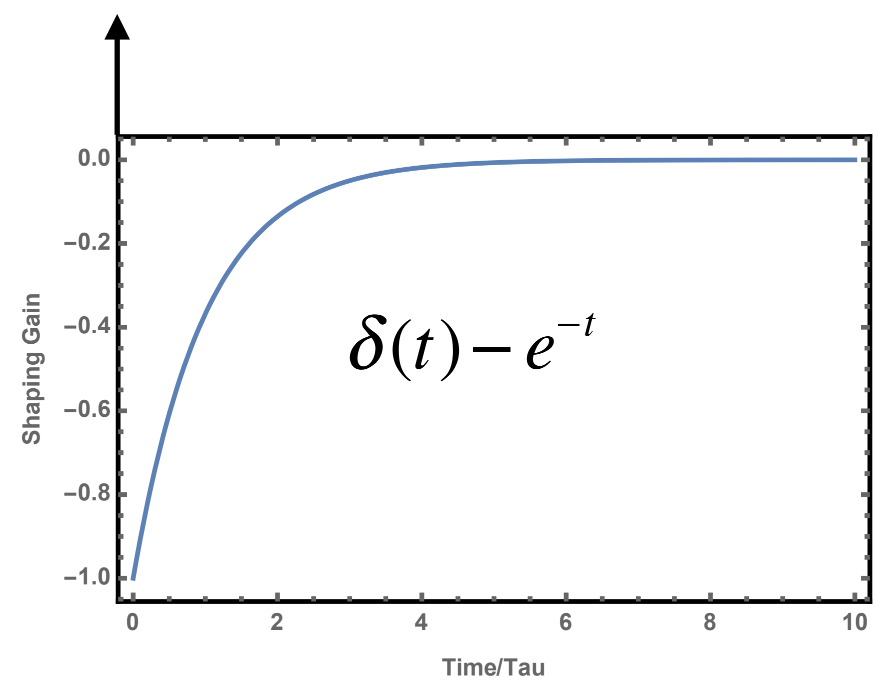
Notice that V_o could just keep increasing if V_i is constant $\Rightarrow$ integrator.

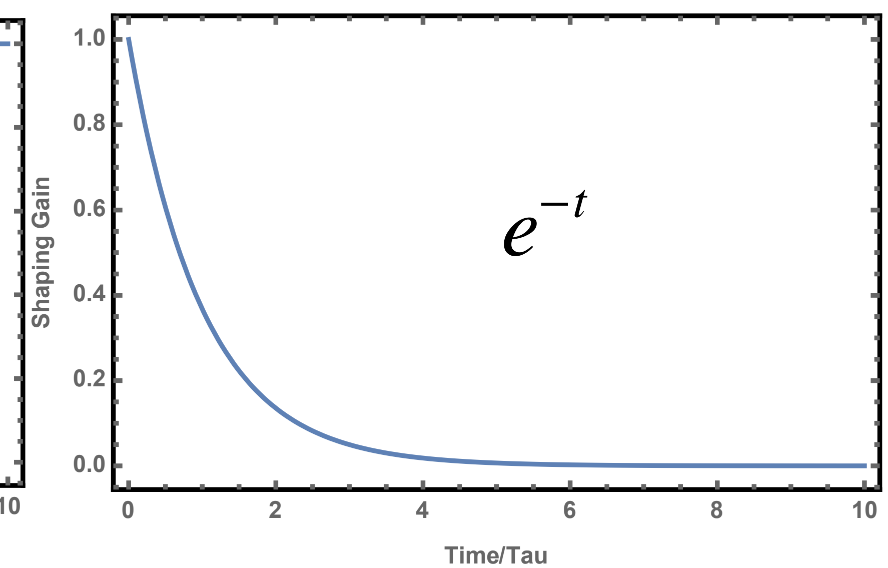
Set RC = 1

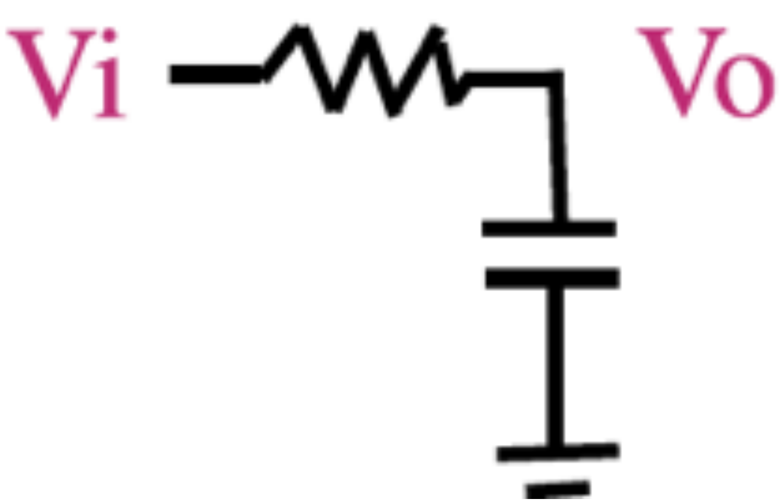
Output if Input is delta(t)

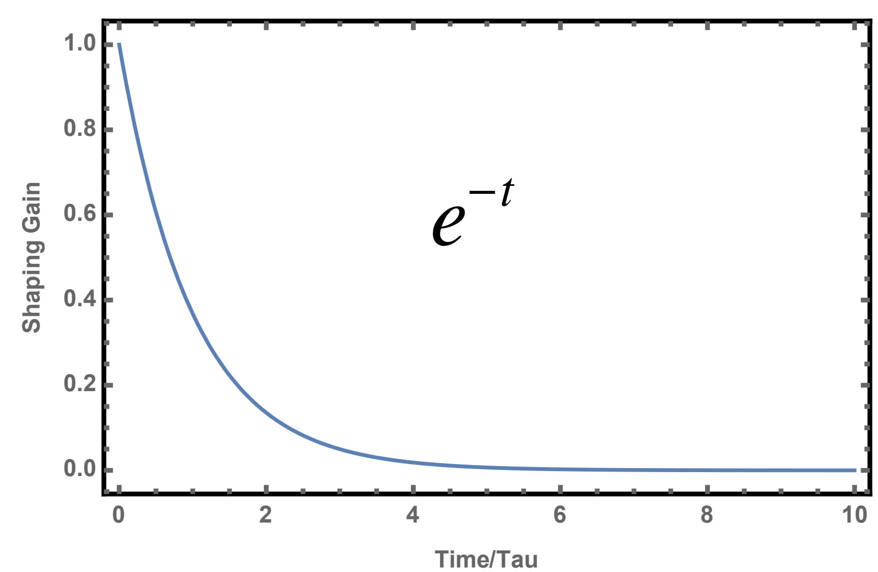
Output if Input is Step function u(t) = 1 for t>0

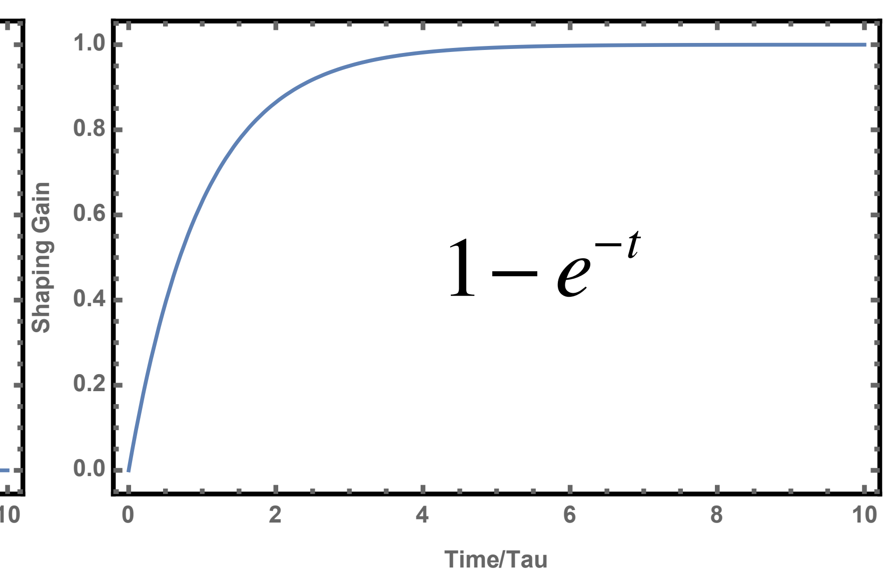




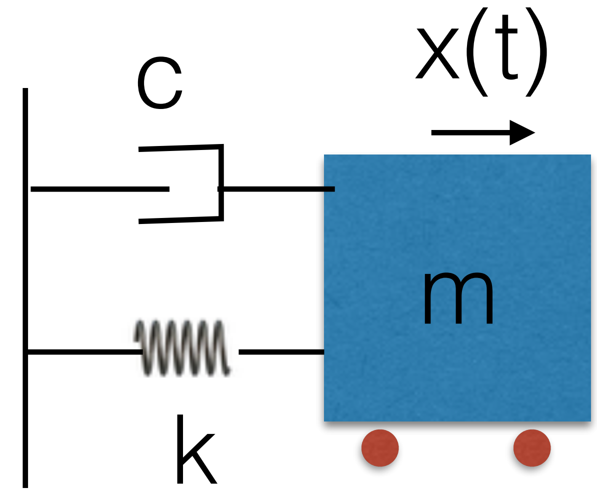








Simple 1 DOF system



$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

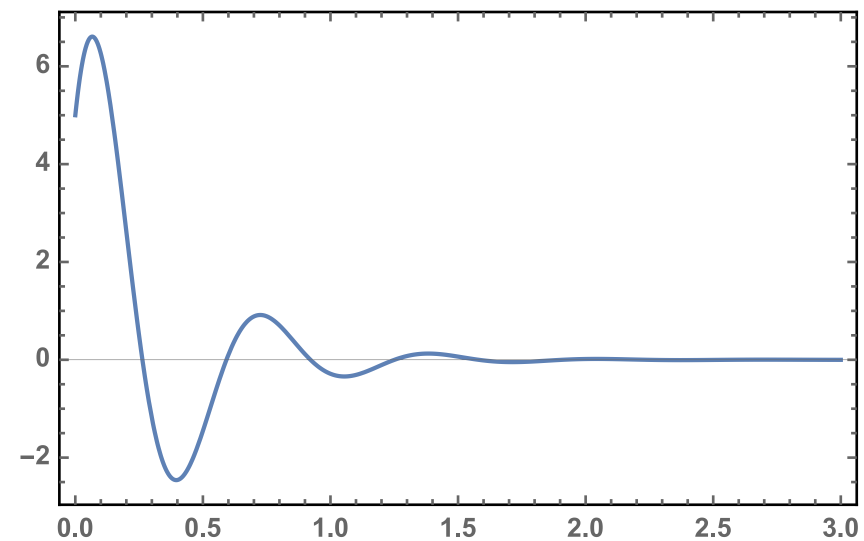
Set $x(t)=Xe^{i\omega t}$

$-\omega^2 mX+i\omega cX+kX=0$ This yields a solution for ω

$$\omega=\text{-}i\frac{c}{2m}\pm\sqrt{\frac{k}{m}}\sqrt{1-\frac{c^2}{4mk}}$$

Set $\zeta=\frac{c}{2m\omega_n}$, $\omega_n^2=\frac{k}{m}$ is the natural frequency

$\omega=-i\zeta\omega_n\pm\omega_n\sqrt{1-\zeta^2}$, the complete solution is then

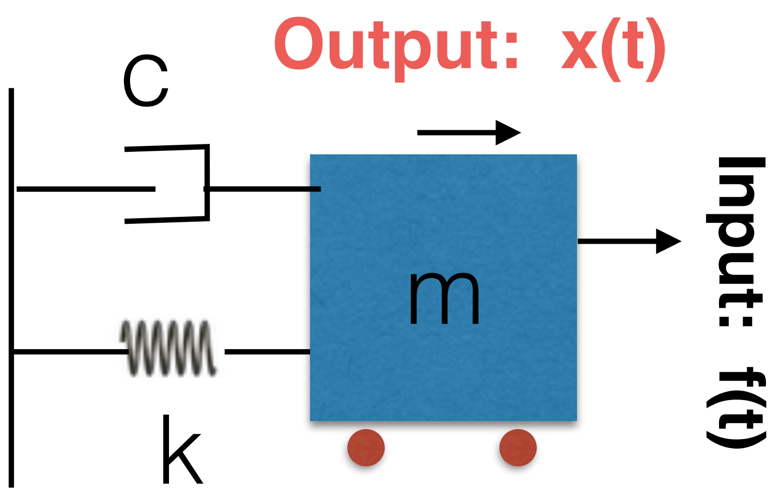


$$x(t)=e^{-\xi\omega_n t}(x_0\,\mathrm{Cos}(\omega_d t)+\frac{v_0+\xi\omega_n x_0}{\omega_d}\mathrm{Sin}(\omega_d t))$$

where $\omega_d=\omega_n\sqrt{1-\zeta^2}$ is called the damped frequency. At ζ=1 (critical damping), there is no oscillatio
If we assume small damping, then the intercept of this motion is
the initial displacement x_0 and the initial slope corresponds to $\sim v_0$

This is not a forced system and it will respond in case of an initial condition that is non-zero.

Example 3 forced mechanical system



$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f(t)$$

Fourier: $x(t)\Leftrightarrow X(\omega);\ f(t)\Leftrightarrow F(\omega)$

$$-\omega^2 mX+i\omega cX+kX=F(\omega)$$

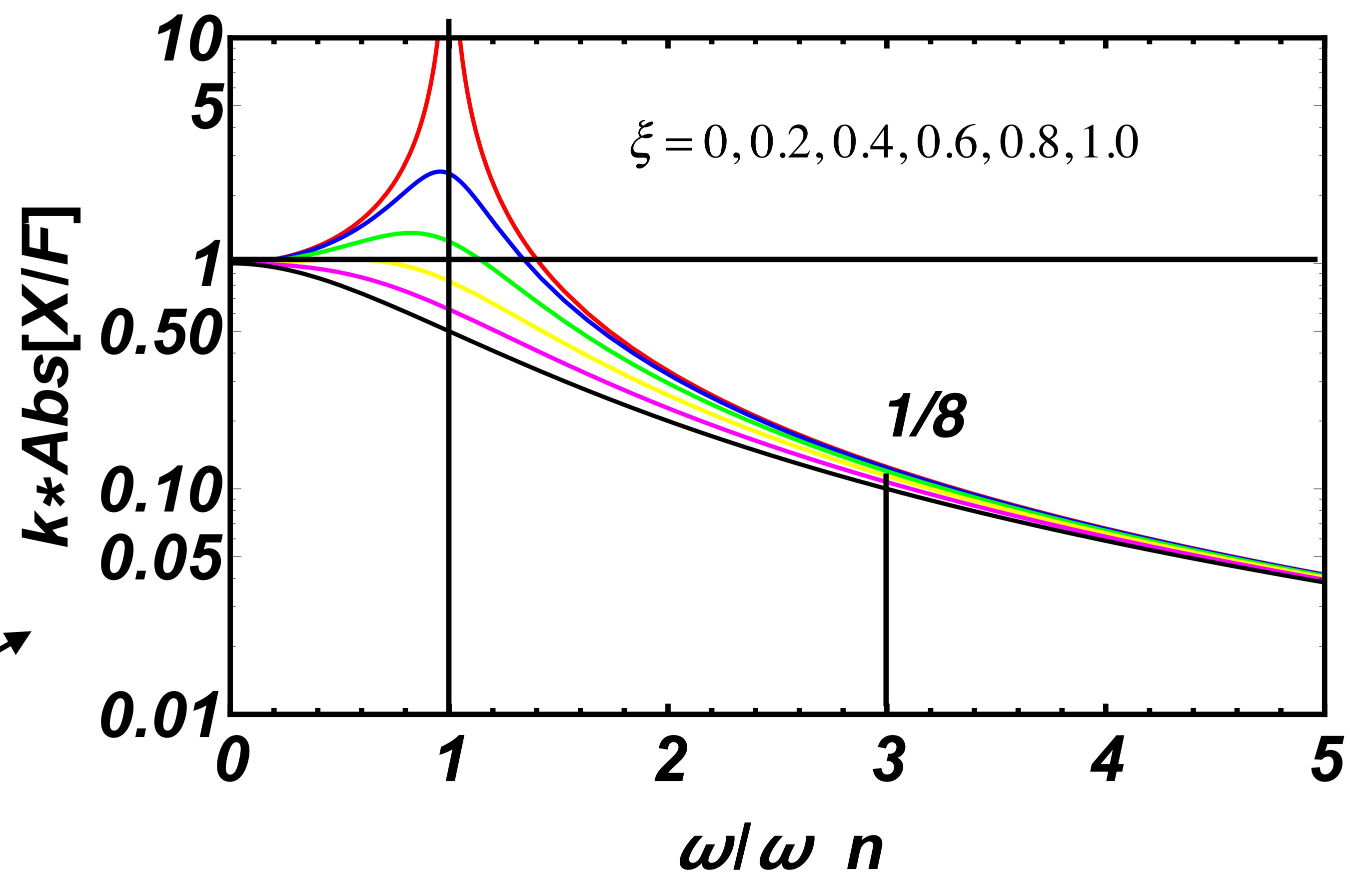
$$\frac{X}{F}=\frac{1}{k}\left[\frac{1}{(1-\omega^2/\omega_n^2)+2i(\omega/\omega_n)\zeta}\right]$$

Units k: N/m, c: N/m/s
f(t): applied force in N

$$\omega_n^2=\frac{k}{m}\qquad \zeta=\frac{c}{2\omega_n m}$$

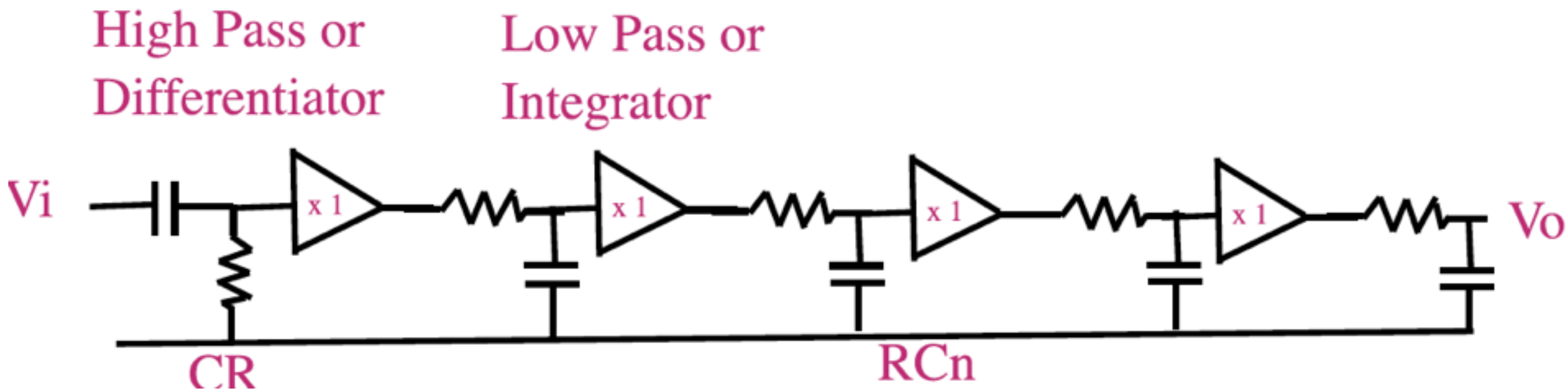
$$\text{Natural Freq}=\nu_n=\frac{\omega_n}{2\pi}\quad \text{Hz}$$

Damping $=\zeta$ is unitless

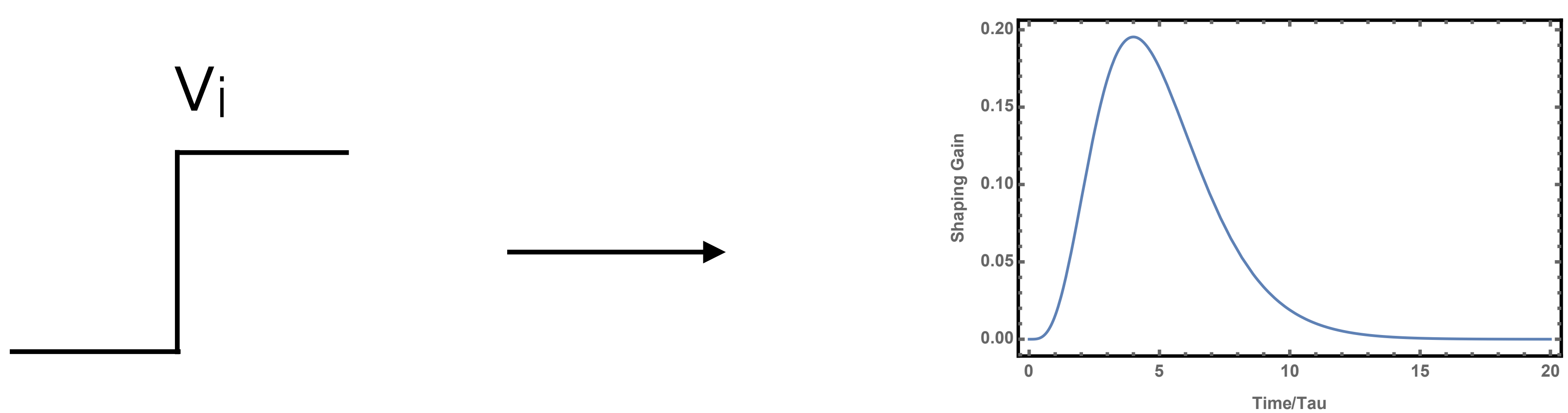


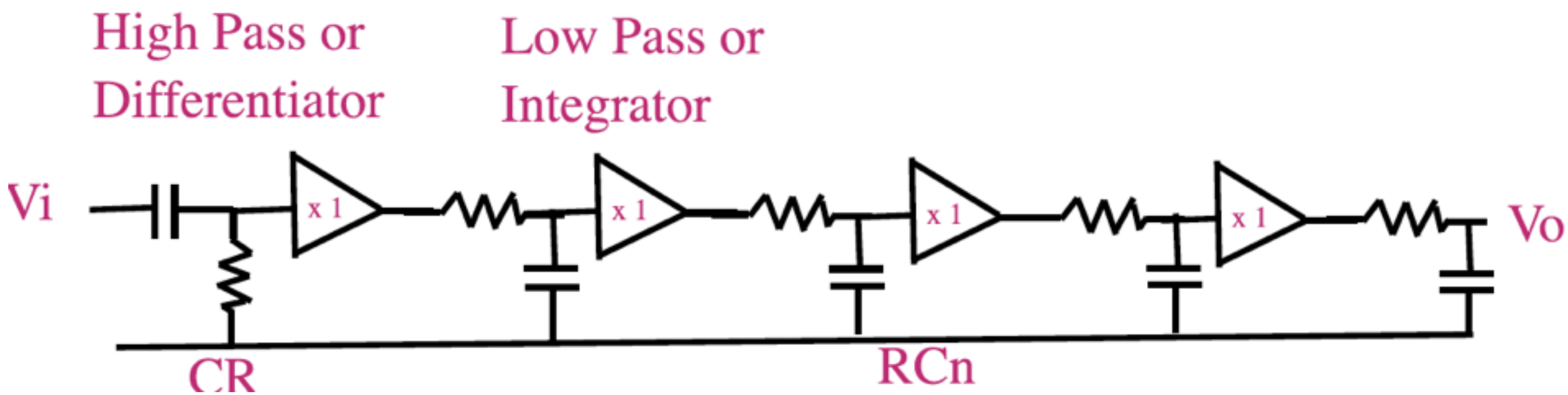
This is called the transfer function. The height of the response at the natural frequency depends on the damping.

Example 4: CR-RC⁴ filter



Purpose is to create a Gaussian shaped pulse from an initial step voltage. The height of the pulse should be the voltage step. The peak of the pulse is given by the peaking time which is n τ=RC = 1/γ





CR-RC⁴ we can just multiply the transfer functions to get the complete transfer function. Using Laplace transforms we get. (is this case s -> iω to get Fourier)

$$V_o(s) = V_i(s) \times \frac{s}{s + 1/\tau} \times \frac{1}{(s + 1/\tau)^4}$$

This can be inverted to obtain time domain pulse for some V_i. For a unit step pulse →

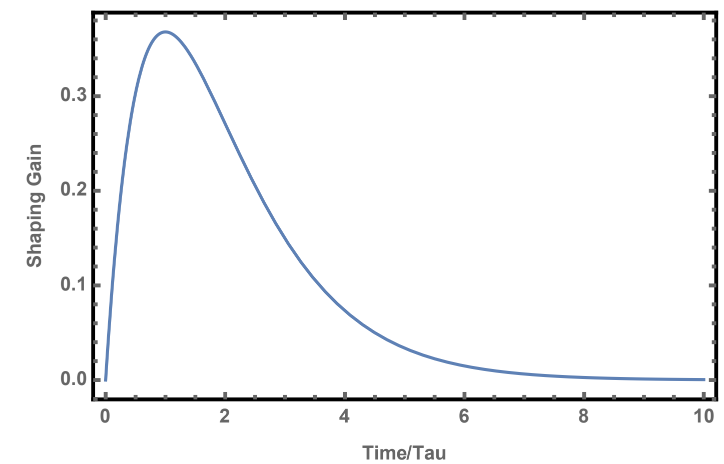
$$V_i(s) = 1/s$$

There are ways of making this more symmetric by introducing complex "poles" in the transfer function.

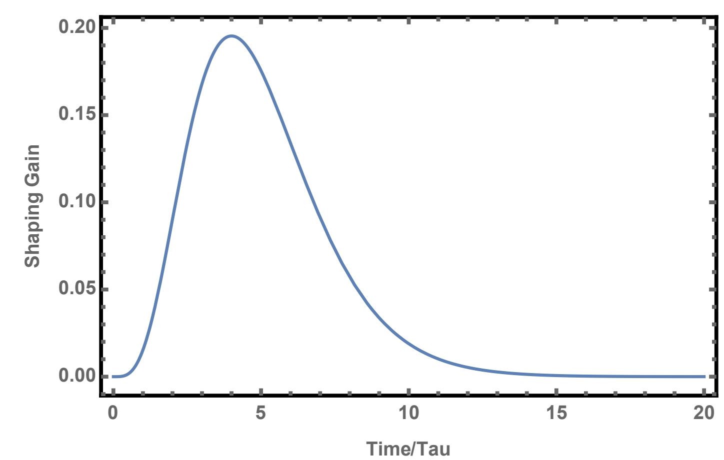
$$V_o(t) = \frac{t^4}{4!} e^{-t/\tau}$$

The ideal preamp produces a step function called a "tail pulse". This step must be shaped.

Input is step function
with a CR-RC filter.
Peak is at time =1*τ

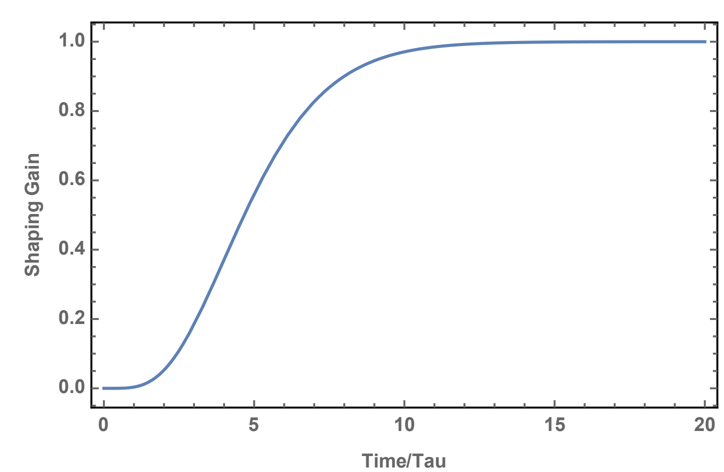


Input is step function
with a CR-RC^4 filter.
Peak is at time =4*τ



delta response will have an undershoot

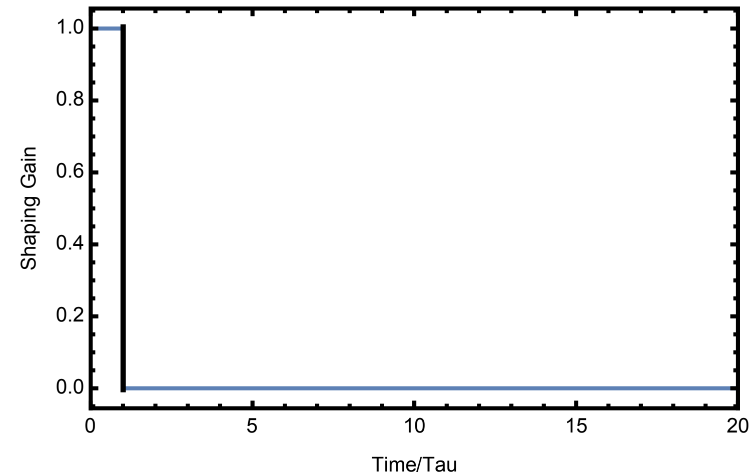
Input is step function
with a RC^5 integrator.



work out the response to a square pulse

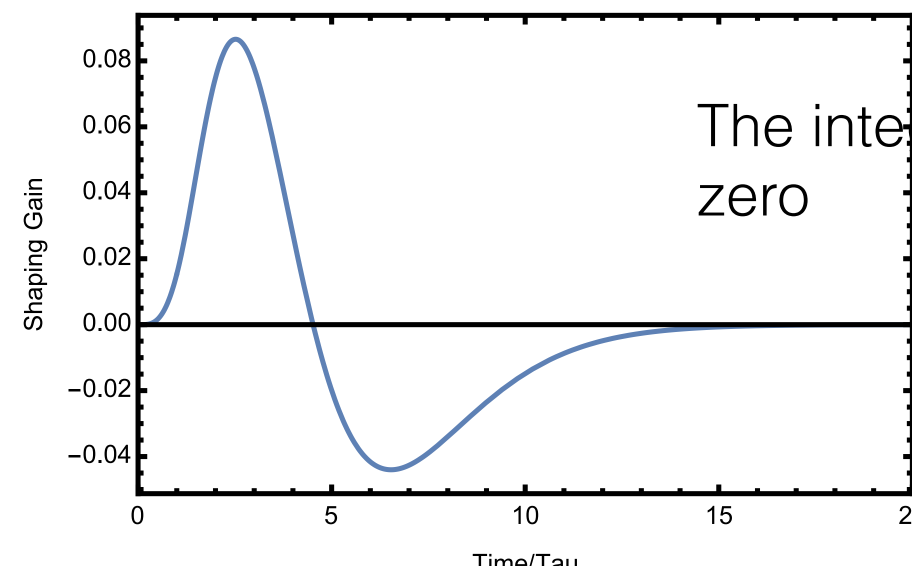
The real input pulses are pulses with some widths. Or they have a long shaping time to bring them back to baseline.

Input



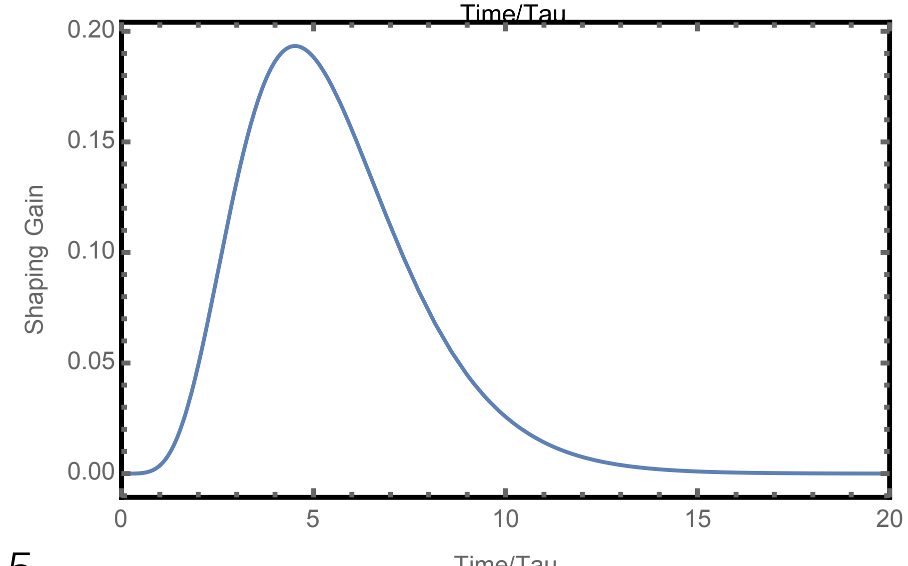
1 mu-sec square pulse

Output after CR-RC4



The integral will be zero

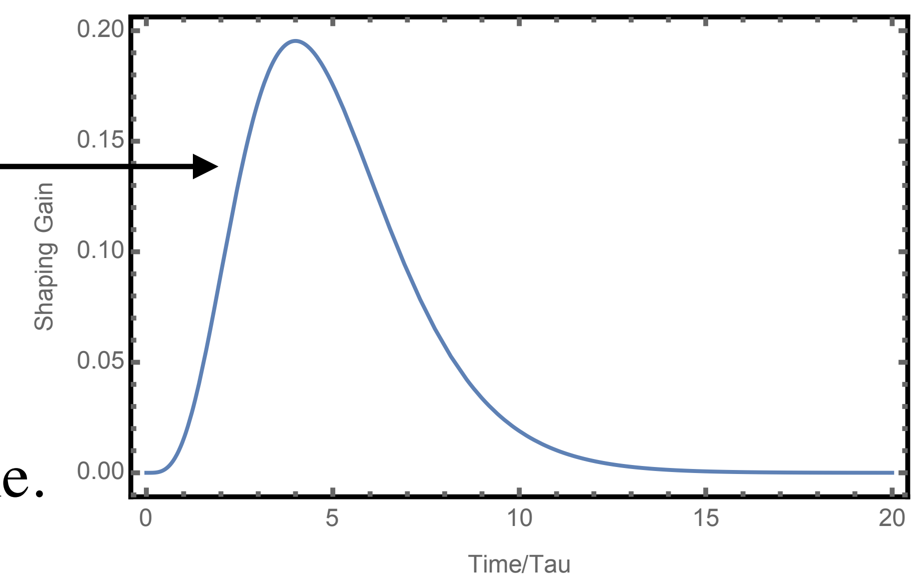
Output after RC5



More about shaping, let's start with a 5th order shaper

$$F(s)=\frac{1}{(s+a)^5}\rightarrow f(t)=\frac{t^4}{4!}e^{-at}$$

This has a maximum at $t=4/a$



$$f(4/a)=\frac{32}{3}\frac{e^{-4}}{a^4}=\frac{0.195}{a^4}$$

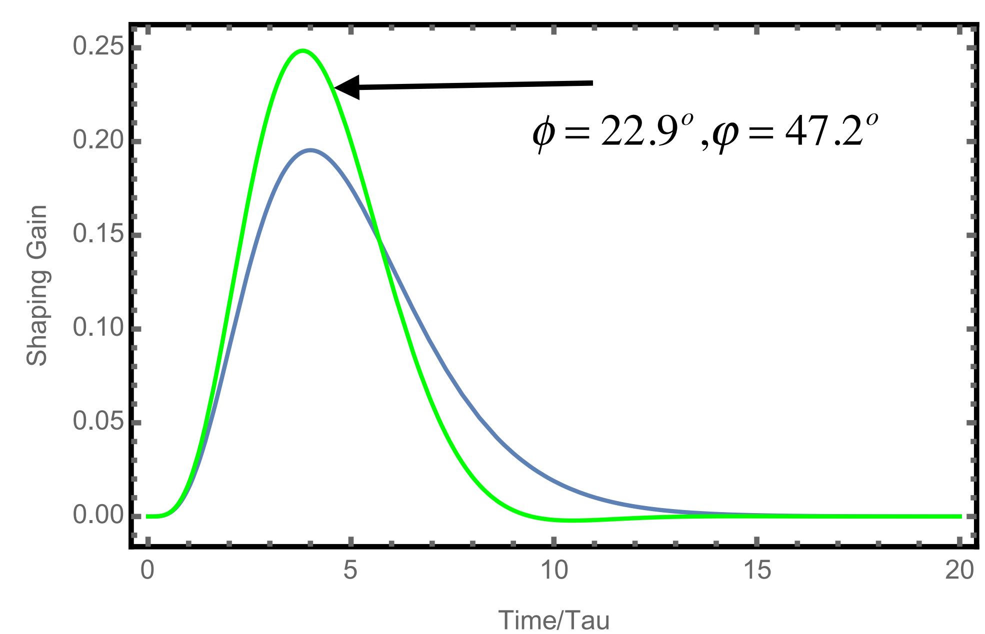
$f(15/a)\approx 0.6*10^{-3}\cdots$It takes 15 times to restore baseline.

Complex poles allow the baseline to be restored much faster.

$$F(s)=\frac{1}{(s+a)((s+a\cos(\phi))^2+a^2\sin^2(\phi))((s+a\cos(\varphi))^2+a^2\sin^2(\varphi))}$$

$$\rightarrow f(t)=Ae^{-at}+\sum_{i=2,3}B_i e^{-r_i t}\cos(c_i t+\gamma_i)$$

In addition, we can adjust the amplitude to obtain the same peak for any value of shaping time.



Ohkawa,Yoshizawa,Husimi, NIM 138, 85-92, 1976

If you want to explore more….(also look at control theory)

There is an extensive theory behind optimum shaping of analog pulses using either analog or digital methods. Digital shaping can be done by gated integration. Shaping using analog electronics is called time-invariant shaping. It uses a suitable configuration of poles to create a semi-Gaussian output. Examples of unipolar shaping are

$$T(s) = \frac{1}{(s+p)^n} \quad \text{r-shapers } n = 2,3,4...$$

$$T(s) = \frac{1}{\prod_{i=1}^{n/2} [(s+r_i)^2 + c_i^2]} \quad \text{c-shapers } n = 2,4,6...$$

r_i and c_i are real and imaginary parts of complex-conjugate poles.

Time domain pulses can be obtained by using partial fraction expansion and inverse Laplace

$$f(t) = \frac{1}{(n-1)!} t^{n-1} e^{-tp} \quad \text{this is for r-shapers } n = 2,3,4,...$$

$$f(t) = \sum_{i=1}^{n/2} 2\left|K_i\right| e^{-r_i t} \operatorname{Cos}(-c_i t + Arg(K_i)) \quad \text{for c-shapers } n = 2,4,6...$$

K_i are obtained from partial fraction expansion. As n increases this becomes more Gaussian.

The peaking time (τ_p) characterizes the frequencies that are filtered and the noise performance.

The width (τ_w) or time to baseline defines the rate capability. For a given τ_p, the higher the order, the shorter the τ_w

Noise waveforms at the output

- ***Systems will produce random outputs in response to random fluctuations at the input or in internal components. These fluctuations could be due to thermal motions, statistical fluctuations in electrical currents, or environmental disturbances.***

- ***The waveforms can be thought of as continuous variables of time or they could be digitized at discrete intervals.***

- ***How do we categorize and analyze these noise waveforms ? Can it be done generally for all systems ?***

- ***We will do this using some mathematical devices. Most important:***

Dirac's delta function and its Fourier transform

δ(t)=0 if t $\neq$ 0 and

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \text{ and } \mathrm{D}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t}\,dt = 1/\sqrt{2\pi}$$

For the inverse transform $\delta(\mathrm{t}) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega$

Some of the derivations here are considered informal because of the use of the delta function as well as the use of discrete Fourier transforms. A more formal way is to use characteristic functions which I will explain later.

Partial bibliography

- *S.O. Rice Bell Syst. Tech. J 23:283-332 (1944) and 24:46-156 (1945).*
- *W. Schottky Ann. Phys. 57:451-567 (1918). W. Schottky Phys. Rev. 28 (1926).*
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- *statlect.com digital textbook on probability theory*
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- *For an introduction to the method of characteristic functions see: Matthews and Walker, pub. W.A.Benjamin inc. (1970), page 385.*
- *A. Pullia and S. Riboldi, IEEE transactions on nuclear science 51(4), 2004.*

Analysis of noise is an old subject that is still evolving. For deeper understanding, need familiarity with: 1) real and complex analysis, 2) probability theory, 3) calculus of randomness.

There is a lot of other good material. But some of it is confusing. Confusion has origins in either the material being too dated with old definitions or having sloppy mistakes in units of Fourier transforms and power spectra (as pointed out by Millotti).

White noise

Take time interval $0 \rightarrow T$

Assume there are N elementary noise excitations in this time interval.

Each characterized by q_i

$$e(t) = \sum_{i=1}^{N} q_i \delta(t - t_i)$$

If each elementary excitation produces a response $g(t)$ (with Fourier transform G(ω))

then the noise waveform is given by

$$eg(t) = \sum_{i=1}^{N} q_i g(t - t_i)$$

Fourier transform of this is given by

$$EG(\omega) = \sum_{i=1}^{N} q_i G(\omega) e^{-i\omega t_i}$$

We have to examine this function as $N \rightarrow \infty$

Properties of the Fourier transform of a real function

Let $g(t)$ be a real function.

$$G(\omega)=\int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,dt \qquad G^*(\omega)=\int_{-\infty}^{\infty} g(t)e^{-i(-\omega)t}\,dt$$

This is the asymmetric convention

Therefore $G^*(\omega)=G(-\omega)$... is Hermitian

Examine the symmetry of $g(t)$ based on $G(\omega)=|G|e^{iArg(G)}$

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{+i\omega t}\,d\omega$$

$$g(t)=\frac{1}{\sqrt{2\pi}}\left(\int_{0}^{\infty} G(\omega)e^{+i\omega t}\,d\omega+\int_{-\infty}^{0} G(\omega)e^{+i\omega t}\,d\omega\right)$$

$$g(t)=\frac{1}{2\pi}\left(\int_{0}^{\infty} G(\omega)e^{+i\omega t}\,d\omega+\int_{\infty}^{0} G(-\omega)e^{-i\omega t}\,(-d\omega)\right)$$

$$g(t)=\frac{1}{2\pi}\left(\int_{0}^{\infty} G(\omega)e^{+i\omega t}\,d\omega+\int_{0}^{\infty} G^*(\omega)e^{-i\omega t}\,d\omega\right)$$

$$g(t)=\frac{1}{2\pi}\int_{0}^{\infty}(G(\omega)e^{+i\omega t}+G^*(\omega)e^{-i\omega t})\,d\omega$$

$$g(t)=\frac{1}{2\pi}\int_{0}^{\infty}|G(\omega)|\,2\cos(\omega t+Arg(G(\omega)))\,d\omega$$

$|G(\omega)|^2$ is called the power spectrum.

if g(t) is in volts and we imagine it is applied across 1 Ω resistance, then $|G(\omega)|^2$ is the amount of power per unit frequency

Notice that if G(ω) is a real function then g(t) is a symmetric function.

g(t) = g(-t) iff G(ω) is real.

Definitions and units

$g(t)$ is a real function.

A symmetric form of Fourier and Inverse Fourier transforms.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt; \quad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega$$

Fourier transform

If units of $g(t)$ are volts then $G(\omega)$ has units of volts/Hz

set time domain 0 to T with M samples.

$$\Delta = \frac{T}{M}; \text{ define index } k = 0, \ldots, M-1; \quad t_k = \Delta \bullet k; \; g_k = g(t_k)$$

Use asymmetric form of Discrete Fourier Transform

$$G_l = \sum_{k=0}^{M-1} g_k e^{-i2\pi \frac{l \cdot k}{M}}; \quad g_k = \frac{1}{M} \sum_{l=0}^{M-1} G_l e^{i2\pi \frac{l \cdot k}{M}}$$

Discrete Fourier transform

g_k and G_l have the same units. What is the relation between $G(\omega)$ and G_l?

$$G(2\pi f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi f t}\, dt \quad \ldots f = \frac{l}{N \cdot \Delta}$$

$$G(2\pi f_l) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi \frac{l}{N} \frac{t}{\Delta}}\, dt \Rightarrow \tilde{G}_l = \sum_{-M/2}^{M/2} g_k e^{-i2\pi \frac{l \cdot k}{N}} \Delta$$

$$G_l = \tilde{G}_l \cdot \frac{1}{\Delta}$$

It is useful to pay attention to the units when plotting G_l

If the normalization is chosen to be symmetric, then ratio is $\left(\sqrt{\frac{2\pi}{M}}.1/\Delta\right)$

some more simple observations about the DFT

$x_0,\ldots,x_{M-1}$ are real numbers. Imagine it is a waveform.

$$X_l = \sum_{k=0}^{M-1} x_k e^{-i2\pi k\, l/M}$$

$$x_k = \frac{1}{M}\sum_{l=0}^{M-1} X_l e^{i2\pi k\, l/M}$$

$$U_{lk} = \frac{\left(e^{-i2\pi/M}\right)^{l\bullet k}}{\sqrt{M}}, l,k = 0,\ldots M-1,$$ is a unitary matrix, and the DFT is a linear matrix transform

Both of these are M-periodic: $X_{l+M} = X_l$, $x_{k+M} = x_k$

using M-periodicity and that x_k are real: $X_{-l} = X_l^* = X_{M-l}$

Take the case of M to be even:

$X_0 \in \mathrm{Real}$ and $X_{M/2} \in \mathrm{Real}$;

X_1 to $X_{M/2-1}$ are complex and X_{M-1} to $X_{M/2+1}$ are conjugate.

This means there are only $(M/2-1)2+2 = M$ independent numbers.

Take the case of M to be odd:

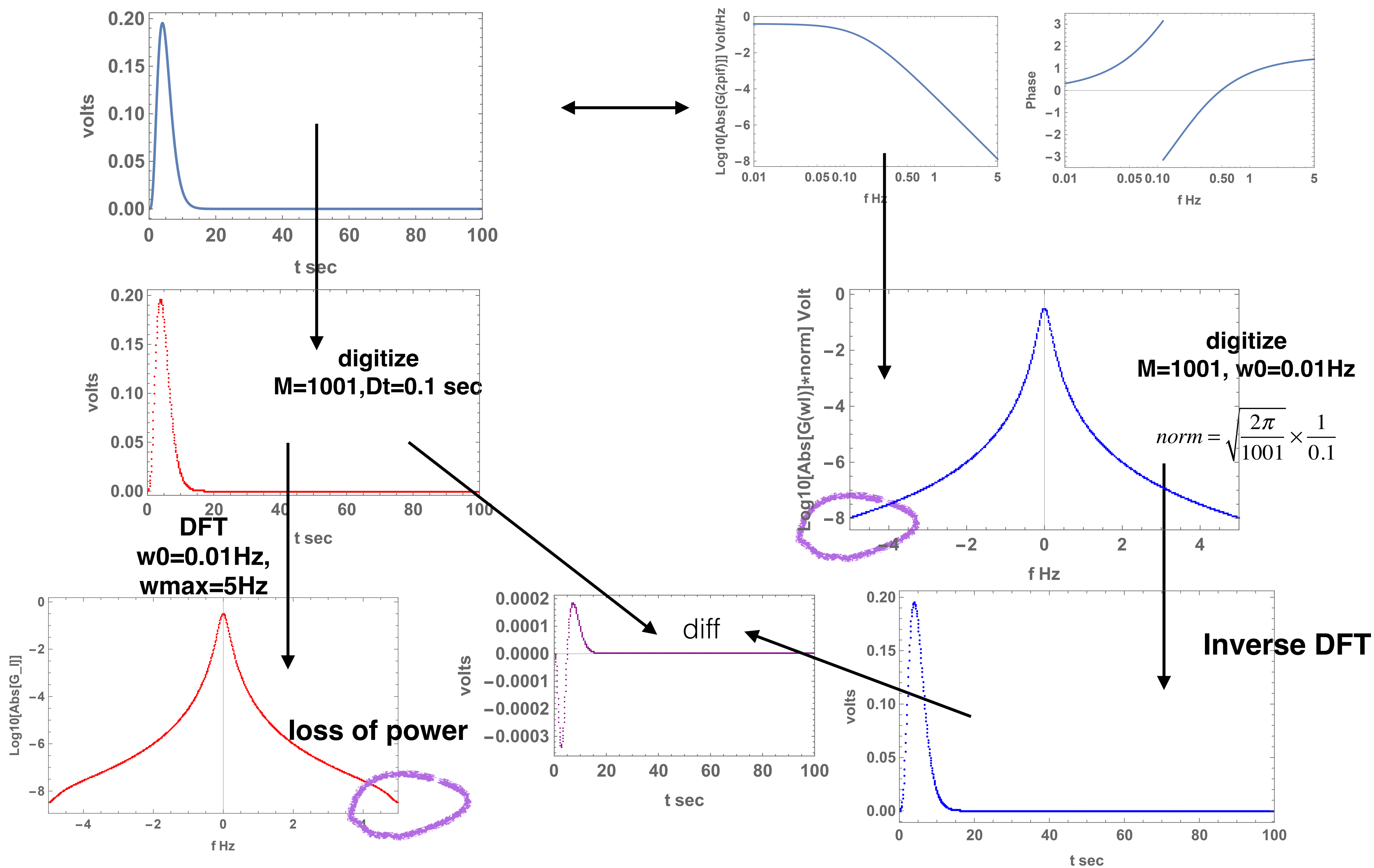
$X_0 \in \mathrm{Real}$

X_1 to $X_{(M-1)/2}$ are complex and X_{M-1} to $X_{(M+1)/2}$ are conjugate.

This means there are only $(M-1)/2\times 2+1 = M$ independent numbers.

non-trivial example (I used symmetric version)

$$g(t) = \frac{e^{-t}t^4}{4!}u(t) \Leftrightarrow G(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1}{(i\omega+1)^5} \quad \text{here } u(t) = \begin{cases} 1 \; if \;\; t \geq 0 \\ 0 \;\; otherwise \end{cases}$$



Well-known characteristics of the noise waveform

Let $eg(t)$ be the noise output (voltage) at time t. This is a real random number at time t.

$$eg(t) = \sum_{i=1}^{N} q_i g(t - t_i) \Rightarrow EG(\omega) = \sum_{i=1}^{N} q_i G(\omega) e^{-i\omega t_i}$$

1) Probability Density Function $P(eg(t))$ is independent of t. (stationarity).

2) The joint probability density of $P(eg(t),\ eg(t'))$ only depends on $(t - t')$

3) The mean $\langle eg(t) \rangle = 0$

It is well known that $eg(t)$ has a Gaussian distribution (we will show this)

These items are needed to specify all characteristics of $eg(t)$.

The variance is given by $e_{rms} = \left\langle eg(t)^2 \right\rangle = P$

$ac(\tau) = \langle eg(t).eg(t+\tau) \rangle$...Autocorrelation.

We will show that the $ac(t)$ is simply the inverse Fourier transform of the power spectrum.

In actuality, one can write down multi-point autocorrelation

$\mathrm{ac}_n(\tau_1, \tau_2, ..., \tau_n) = \langle eg(t) eg(t+\tau_1) ... eg(t+\tau_n) \rangle$

For a stationary process all of these must not depend on t.

Characteristic function method

First we will describe the method of characteristic functions.

A characteristic function is a Fourier transform of a probability density function (PDF).

It makes combinations of probabilities easier to calculate and understand.

X is a continuous random variable with probability density function $P(x)$ then the characteristic function is

$$\varphi_X(k) = \int_{-\infty}^{\infty} P(x) e^{ikx}\, dx$$

This allows easy way to generate moments of the PDF.

$\varphi(k=0)=1$ since it is the integral of the PDF.

$$\langle x \rangle = -i \frac{\partial \varphi}{\partial k}(k=0)$$

$$\langle x^2 \rangle = -\frac{\partial^2 \varphi}{\partial k_i \partial k_j}(k=0)$$

If X and Y are two random variables and $z = f(x,y)$ then the Characteristic function for Z is

$$\varphi_Z(k) = \iint e^{ikf(x,y)} P(x)dx\, Q(y)dy$$

To get the moments of $f(x,y)$ often it is not necessary to evaluate the integral.

e.g. $f(x,y) = x+y \Rightarrow \varphi_Z(k) = \varphi_X(k).\varphi_Y(k)$ .... leave it for you to prove this

Aside on Gaussian PDF and its characteristic func.

Characteristic function makes combinations of probabilities easier to calculate and understand.

$P(x_1, x_2, ..., x_n) = Ne^{-\sum_{i \le j}^{n} a_{ij} x_i x_j}$ is a Gaussian multivariate PDF with a mean of 0 for all x.

$N = \frac{(2 \cdot Det[a_{ij}])^{-1/2}}{(2\pi)^{n/2}}$ is the normalization. The matrix a_{ij} has to be positive definite.

let $X = \{x_1, x_2, ...\}$ and $K = \{k_1, k_2, ...\}$ for short-hand.

$\varphi(k_1, k_2, ..., k_n) = \int_{-\infty}^{\infty} dX \cdot P(X) e^{iK \cdot X} = e^{-\sum_{i \le j}^{n} b_{ij} k_i k_j}$ is the characteristic function

(The Fourier transform of a Gaussian yields a Gaussian; $\mathrm{b}_{ij} = \frac{[a_{ij}]^{-1}}{2}$)

$\varphi(K = 0) = 1$ since it is the integral of the PDF.

$-i \frac{\partial \varphi}{\partial k_i}(K = 0) = \langle x_i \rangle = 0$

$-\frac{\partial^2 \varphi}{\partial k_i \partial k_j}(K = 0) = \langle x_i x_j \rangle = b_{ij}$ for $i \ne j$ and $\langle x_i^2 \rangle = 2b_{ii}$ for $i = j$

The first and second moments are obtained by differentiation

Homework: work out a method for simulating an n-dim Gaussian variate

Aside on Gaussian noise

Now assume that the noise waveform $eg(t)$ is Gaussian.

$\langle eg(t)\rangle = 0$

$\left\langle eg(t)^2 \right\rangle = P$ is the mean square noise amplitude (or the total noise power.)

$ac(\tau) \equiv \langle eg(t)\cdot eg(t+\tau)\rangle$ is called the autocorrelation

Now for $x_1 = eg(t),\ x_2 = eg(t+\tau_2),\ x_3 = eg(t+\tau_3) \ldots x_n = eg(t+\tau_n)$ we can write a joint probability density function. It will have a characteristic function

$$\varphi(k_1, k_2, \ldots k_n) = Exp(-\sum_{i \le j=1}^{n} b_{ij} k_i k_j)$$

Notice that all

$b_{ii} = P/2$ and $b_{ij} = \left\langle eg(t+\tau_i) eg(t+\tau_j)\right\rangle = ac(\tau_i - \tau_j)$

This means that correlations of any order only depend on the total power and the autocorrelation function (which we will prove later to be the power spectrum density). The general rule for obtaining any moment is

$$\left\langle x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \right\rangle = (-i)^{\alpha_1+\alpha_2+\ldots+\alpha_n} \left. \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n} \varphi(k)}{\partial k_1^{\alpha_1}\, \partial k_2^{\alpha_2} \ldots \partial k_n^{\alpha_n}} \right|_{k=0}$$

In the Gaussian case only the total power and the autocorrelation is needed to completely specify φ(K).

Finish Part 1

- ***Linear systems are characterized by a response function which, in general, has a complex Fourier transform.***

- ***This response function is important to understand the random noise waveform that may come from the system.***

- ***In part 2 we will review how the random noise tends to Gaussian (central limit theorem).***

- ***And examine classifications of random noise in terms of power spectra.***